- Perron's theorem can be extended to some non-negative matrices.
- Perron's theorem conclusion is not valid for all non-negative matrices.

Example....

We now introduce non-negative irreducible matrices.

Our aim is to show the following theorem:

Theorem

Let $A \geq 0$ be an irreducible matrix of order n, $n > 1$. Then the following are true:

- $\rho(A)$ *is an eigenvalue.*
- *There is a positive eigenvector of A associated with the eigenvalue* $\rho(A)$.
- *No non-negative eigenvector is associated with any other eigenvalue of A.*
- $\rho(A)$ *is simple.*

Let A be irreducibe, $n \times n$ and non-negative.

Then, $(I + A)^{n-1} > 0$.

Put $\rho := \rho(A)$ and $R :=$ spectral radius$(I + A)^{n-1}$

We now show that

$$(1 + \rho)^{n-1} \leq R.$$

Note: If ρ is an eigenvalue of A, then this is immediately true.

But we don't know this.

So, some argument is needed.

Step 1 will be to show the above inequality.

- Let $Ax = \lambda x$ and $x \neq 0$.
- $m(Ax) = m(\lambda x) = |\lambda| m(x)$.
- Consider λ for which $|\lambda| = \rho$.

We already know that

$$m(Ax) \leq A m(x).$$

Use all these inequalities + $m(Ax) \geq 0$.

$$m(Ax) = |\lambda| m(x) = \rho m(x) \leq A m(x).$$

What we need is:

$$\rho m(x) \leq A m(x).$$

Multiply by ρ.

$$\begin{aligned} \rho^2 m(x) &\leq \rho A m(x) \\ &= A \rho m(x) \\ &\leq A^2 m(x) \end{aligned}$$

In general for any k,

$$\rho^k m(x) \leq A^k m(x).$$

It follows that

$$(1+\rho)^{n-1} m(x) \leq (I+A)^{n-1} m(x). \quad \textit{Why?} \tag{1}$$

Apply Perron's theorem to the transpose of $(I+A)^{n-1}$.

There exists $y > 0$ such that

$$y^T (I+A)^{n-1} = R y^T, \quad y > 0.$$

Hence by (1),

$$(1+\rho)^{n-1} y^T m(x) \leq R y^T m(x).$$

- $y^T m(x) > 0$.

So,

$$(1+\rho)^{n-1} \leq R.$$

This completes step 1.

Use the above inequality to show that ρ is an eigenvalue of A. This is step 2.

Let the eigenvalues of A be $\alpha_1, \alpha_2, \ldots, \alpha_n$.

Then, the eigenvalues of $(I + A)^{n-1}$ are:

$$(1 + \alpha_1)^{n-1}, (1 + \alpha_2)^{n-1}, \ldots$$

$(I + A)^{n-1}$ is positive.

By Perron's theorem, spectral radius of $(I + A)^{n-1}$ is an eigenvalue of $(I + A)^{n-1}$.

For some k, $R = |(1+\alpha_k)^{n-1}|$.

Put $\mu = \alpha_k$.

From step (1),

$$(1+\rho)^{n-1} \leq R.$$

This means that:

$$(1+\rho)^{n-1} \leq |(1+\mu)^{n-1}|.$$

Taking the $(n-1)$-th root on both the sides gives:

$$1+\rho \leq |1+\mu|.$$

Also,

$$|1+\mu| \leq 1+|\mu|$$

$\mu = \alpha_k$ is one of the eigenvalues of A.

So, $|\mu| \leq \rho(A)$.

Now write all the inequalities:

$$1+\rho \leq |1+\mu| \leq 1+|\mu| \leq 1+\rho.$$

Hence,

$$|1+\mu| = 1+|\mu|.$$

This gives $\mu \geq 0$ and $\rho \leq |\mu|$.

Hence, $\mu = \rho$.

Thus, spectral radius is an eigenvalue of an irreducible non-negative matrix.

The first assertion is complete.

We now prove the following assertion of the Frobenius theorem:

If $x \in \mathbb{R}^n$ is an eigenvector of A associated with ρ, then we show the following:

- $m(x)$ is an eigenvector of A.
- $m(x) > 0$.

To complete the proof it is very simple.

Just go back to the proof of the first assertion.

- Let $Ax = \rho x$ and $x \neq 0$.
- $m(Ax) = m(\rho x) = \rho m(x)$.

We already know that

$$m(Ax) \leq Am(x).$$

Suppose $m(Ax) = Am(x)$. Then,

$$Am(x) = \rho m(x).$$

So, $m(x)$ will be an eigenvector of A corresponding to the spectral radius.

Difficulty arises only when the inequality

$$m(Ax) \leq A(m(x))$$

is strict.

We now get a contradiction by assuming that

$$m(Ax) < A(m(x)).$$

But,

$$m(Ax) = m(\rho x) = \rho m(x).$$

So,

$$m(Ax) < A(m(x))$$

implies that

$$\rho m(x) < Am(x).$$

$$\begin{aligned}\rho^2 m(x) &< \rho Am(x)\\ &= A(\rho m(x))\\ &< A(Am(x)) = A^2 m(x).\end{aligned}$$

In general for any k,

$$\rho^k m(x) < A^k m(x).$$

If f is any polynomial with positive coefficients, then

$$f(\rho)m(x) < f(A)m(x).$$

(Why?)

Define

$$g(\delta) := (1 + \delta)^{n-1}.$$

g is a polynomial with positive coefficients.

So,

$$g(\rho)m(x) \leq g(A)m(x).$$

Hence,

$$(1 + \rho)^{n-1}m(x) < (I + A)^{n-1}m(x).$$

As before, let

$$R := \rho(I + A)^{n-1}.$$

R is the spectral radius of $(I + A)^{n-1}$.

Apply Perron's theorem to $(I + A)^{n-1}$.

There exists $y > 0$ such that

$$y^T (I + A)^{n-1} = R y^T.$$

We have

$$(1+\rho)^{n-1} m(x) < (I+A)^{n-1} m(x).$$

and

$$y^T (I+A)^{n-1} = y^T R, \quad y > 0.$$

Hence

$$(1+\rho)^{n-1} y^T m(x) < R y^T m(x).$$

This gives

$$(1+\rho)^{n-1} < R.$$

By Perron's theorem R must be an eigenvalue of $(I+A)^{n-1}$.

If f is a polynomial, and if λ is an eigenvalue of A, then $f(\lambda)$ is an eigenvalue of $f(A)$.

Denote the eigenvalues of A by $\alpha_1, \ldots, \alpha_n$.

Then the eigenvalues of $g(A)$ are:

$$(1+\alpha_1)^{n-1}, (1+\alpha_2)^{n-1}, \ldots$$

Since R is an eigenvalue of $g(A)$.

$$\beta = |(1+\alpha_k)^{n-1}|$$

for some k.

Set $\mu = \alpha_k$.

We have

$$R = |1 + \mu|^{n-1}.$$

Already we have

$$(1 + \rho)^{n-1} < R.$$

So,

$$(1 + \rho)^{n-1} < |1 + \mu|^{n-1}.$$

This gives

$$1 + \rho < |1 + \mu|.$$

So, $\rho < |\mu|$.

This is a contradiction.

Hence strict inequality in

$$Am(x) \leq m(Ax)$$

is not possible.

So,

$$\begin{aligned} Am(x) &= m(Ax) \\ &= m(\rho x) \\ &= \rho m(x). \end{aligned} \tag{2}$$

Claim: $m(x) > 0$

$A^k m(x) = \rho^k m(x)$.

Recall: $g(\delta) = (1 + \delta)^{n-1}$.

Hence,

$$g(\rho)m(x) = g(A)m(x).$$

But $g(A) > 0$.

$m(x)$ is non-negative.

So, by Perron's theorem, $m(x) > 0$.

We have shown the following:

If A is a non-negative irreducible matrix, then $Ax = \rho x$ for some $x > 0$.

From, the arguments we have seen so far,

$$Ax = \rho x \;\;\Rightarrow\;\; (I + A)^{n-1}x = (1 + \rho)^{n-1}x.$$

Nullity of $A - \rho I$ and Nullity of $(I + A)^{n-1} - \rho I$ must be same.

Thus, geometric multiplicity of ρ is 1.

Let $A \geq 0$ be an irreducible matrix. Then, $\rho(A)$ is called the Perron root of A.

Associated eigenvector is called Perron vector.

Algebraic multiplicity of ρ is one.

(Note: This will imply that geometric multiplicity is one).

Suppose ρ is not a simple eigenvalue of A.

Then, there exist v and w such that

$$Av = \rho v$$

$$Aw = v + \rho w.$$

Exercise

We have $v > 0$.

$y^T(A - I)w = v^T y$

Let y be the left Perron vector of A.

Then, $v^T y = 0$.

But $v^T y$ cannot be 0.

Suppose,

$$Ay = \alpha y, \;\; \alpha \neq \rho, \;\; y \geq 0.$$

Then get a contradiction.

Exercise.

We have proved Frobenius theorem.

What happens for reducible non-negative matrices?

What can be said about principal submatrix of a non-negative matrix?