Principal sub matrix of a nonnegative matrix

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THM: Let $A \geq 0$ be an irreducible matrix. Let

$$A = \begin{bmatrix} M & A_{12} \\ A_{21} & A_{22} \end{bmatrix}.$$

Then $\rho(A) > \rho(M)$.

PROOF: Define $B = \begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix}$.

OBSERVATIONS

1. $A \geq B \geq 0$
2. If $C = \frac{A+B}{2}$, then

$A \geq C \geq B \geq 0$.

3. $C \geq \frac{A}{2} \Rightarrow C$ is irreducible

Now we can complete the proof

Let $C^T u = \beta u$ $u > 0$ $\beta > 0$

$Ay = \lambda_0 y$ $y > 0$ $\lambda_0 > 0$.

Since $A_{12} \geq 0$, $A_{12} \neq 0$.

So,
$$\beta \langle u, y \rangle = \langle C^T u, y \rangle$$
$$= \langle u, Cy \rangle$$
$$< \langle u, Ay \rangle$$
$$= \langle u, \lambda_0 y \rangle$$

$\Rightarrow \beta < \lambda_0$.