Lemma: Suppose A is $n \times n$.
Let λ be an e.v of A. Assume that $G.multiplicity(\lambda) < A.Mult.(\lambda)$.
Then, $\exists$ of $v \in \mathbb{C}^n$ and $w \in \mathbb{C}^n$ such that
$$Av = \lambda v$$
$$Aw = v + \lambda w.$$

PROOF: WLOG, $\lambda = 0$. Let $Ap^1 = 0p^1$; $\|p\| = 1$. ; WLOG $A.M(0) = 2$, $G.M.(0) = 1$

Define $U = [p^1\ p^2\ p^3\ p^4 \dots p^n]$; so that U is unitary.

$$AU = [Ap^1\ Ap^2 \dots Ap^n]$$
$$= [0p^1\ q^2 \dots q^n] \text{ (say)}$$

$$U^* A U = [0, U^* q^2, \dots, U^* q^n]$$
$$= [0, v^2, \dots, v^n]$$

Write
$$U^* A U = \begin{bmatrix} 0 & v^2_1 & \dots & v^n_1 \\ 0 & \boxed{\begin{matrix} v^2_2 & & v^n_2 \\ & & \\ v^2_n & & v^n_n \end{matrix}} \\ \vdots & & & \\ 0 & & & \end{bmatrix}$$

Put $B = \begin{pmatrix} v^2_2 & \dots & v^n_2 \\ \vdots & & \vdots \\ v^2_n & \dots & v^n_n \end{pmatrix}$

Find P such that
$$P^* \begin{pmatrix} v^2_2 & \dots & v^n_2 \\ \vdots & & \\ v^2_n & & v^n_n \end{pmatrix} P = \begin{bmatrix} 0 & \alpha^2_2 & \dots & \alpha^n_2 \\ \vdots & & & \\ 0 & \alpha^2_n & & \alpha^n_n \end{bmatrix}$$

(Since '0' must be an e.value of B.)
$A.M.(A) \geq 2$

Put $\tilde{P} = \begin{bmatrix} 1 & 0 \\ 0 & P \end{bmatrix}$.

$\tilde{P}$ is unitary.

$V = \tilde{P} U$ is unitary.

$$V^* A V = \begin{bmatrix} 0 & \eta & q^1_1 & \dots & q^n_1 \\ 0 & 0 & q^1_2 & \dots & q^n_2 \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & q^1_n & \dots & q^n_n \end{bmatrix}$$

If $\eta = 0$, then
$$V^* A [v^1, v^2, \dots, v^n]$$
$$= [V^* A v^1, V^* A v^2, \dots]$$
$$= [0, 0, \dots]$$
$\Rightarrow Av^1 = 0$, $Av^2 = 0$ (v^1, v^2 are L.I).

$G.M.(0) = 2$
(Contradiction to our assumption)

So, $\eta \neq 0$.

$$V^* A V = [0, \eta e_1, *, \dots]$$
$$(V^* A v^1, V^* A v^2) = (0, \eta e_1)$$
$\Rightarrow$ $Av^1 = 0$
$V^* A v^2 = \eta e_1 \Rightarrow Av^2 = \eta V e_1 = \eta v^1$

$Av^1 = 0$
$Av^2 = \eta v^1$
$\Rightarrow Av^1 = 0$; $A\left(\frac{v^2}{\eta}\right) = v^1$.

$Av^1 = 0v^1$
$A\left(\frac{v^2}{\eta}\right) = 1.v^1 + 0v^2$

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Supp. $G.M.(0) = 2$ and $A.M.(0) = 3$

Then $Av^1 = 0$
$Av^2 = 0v^1 + 0v^2$
$Av^3 = \underbrace{(\alpha_1 v^1 + \alpha_2 v^2)}_{w} + 0v^3$

$\{v^1, v^2, v^3\}$ are orthogonal.

$Av^3 = w + 0v^3$

$$\boxed{\begin{matrix} Aw = 0 \\ Av^3 = 1.w + 0v^3 \end{matrix}}$$

— x —