NON NEGATIVE.

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Non-negative

THM: Let $A \geq 0$. Then

[1] $\rho(A)$ is an e.v. of A.

[2] $Ax = \rho(A)x$ for some $x \geq 0$; $x \neq 0$.

PROOF

FIRST OBSERVATION

Either A is reducible or irreducible.

If A is irreducible, then we know the result.

Assume that A is reducible.

WLOG, assume that

$$A = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix};$$

B & C are square matrices.

By induction, it easily follows that $\rho(A)$ is an e. value.

In fact

$$\rho(A) = \text{MAX}\{\rho(B), \rho(D)\}.$$

Suppose $\rho(A) = \rho(B) = \lambda_0$

Let $By = \lambda_0 y$ $(y \geq 0)$

$$A = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix}$$

$$\begin{bmatrix} B & C \\ 0 & D \end{bmatrix}\begin{bmatrix} y \\ 0 \end{bmatrix} = \begin{bmatrix} \lambda_0 y \\ 0 \end{bmatrix} = \lambda_0 \begin{bmatrix} y \\ 0 \end{bmatrix}.$$

Hence the theorem is proved.

If $\rho(A) = \rho(D)$, then some argument is needed. $\{\rho(A) > \rho(B)\}$

Put $\rho(D) = \beta$.

Let $Dp = \beta p$ $(p \geq 0)$.

$$\begin{bmatrix} B & C \\ 0 & D \end{bmatrix}\begin{bmatrix} w \\ p \end{bmatrix} = \begin{bmatrix} Bw + Cp \\ \beta p \end{bmatrix}$$

We want 'w' such that $Bw + Cp = \beta w$ & $w \geq 0$.

FIRST OBSERVATTON

① $(B - \beta I)$ is a non-singular matrix.

SECOND OBS:

$$\beta I - B = \beta\left(I - \frac{B}{\beta}\right).$$

$$(\beta I - B)^{-1} = \frac{1}{\beta}\left(I - \frac{B}{\beta}\right)^{-1}$$

$$= \frac{1}{\beta}\left(I + \frac{B}{\beta} + \frac{B^2}{\beta^2} + \cdots\right)$$

$$\rho\left(\frac{B}{\beta}\right) = \frac{1}{\beta}\rho(B)$$

< 1 (Since $\rho(B) < \beta = \rho(D) = \rho(A)$)

So, $(\beta I - B)^{-1} \geq 0$.

We now complete the proof.

$$\left((B - \beta)w + Cp = 0 \iff w + (\beta - \beta I)^{-1}Cp = 0.\right)$$

Choose $-w = (\beta - \beta I)^{-1}Cp$

Then $w \geq 0$ (why).

Now compute

$$\begin{bmatrix} B & C \\ 0 & D \end{bmatrix}\begin{bmatrix} w \\ p \end{bmatrix}$$

$$= \begin{pmatrix} Bw + Cp \\ Dp \end{pmatrix} = \begin{pmatrix} \beta w \\ \beta p \end{pmatrix} = \beta\begin{pmatrix} w \\ p \end{pmatrix}.$$

We have proved.

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