Irreducibe matrices- A short introduction

Consider the following matrix:

$$D := \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$

Spectral properties of D:

$$De = 3e.$$

$\widetilde{D} := D\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ is a principal submatrix of D.

$$\det \widetilde{D} = 2.$$

$\widetilde{D}(1, 1, 1)^T = 2(1, 1, 1)$.

$\widetilde{D}$ has exactly one positive eigenvalue.

So, $\widetilde{D}$ has two negative eigenvalues.

By interlacing theorem, D has at least two negative eigenvalues.

$\det D < 0$

So, D has 3 negative eigenvalues.

- $\rho(D) = 3$.
- The eigenvector e is positive.
- Geometric multiplicity of D is 1.
- All the conclusions in Perron's theorem are true for D.
- D is not a positive matrix.
- It is just a non-negative matrix.

But Perron's theorem is not true for any non-negative matrix.

Example....

But for non-negative matrices, at least spectral radius will be an eigenvalue. (We will see this later.) Needs some argument to prove.

We will now prove Perron's theorem to a class of non-negative matrices that have certain form.

These matrices will be called irreducible matrices.

The proof uses Perron's theorem.

Proved by Frobenius.

Naturally, the definition of irreducible matrices, must depend on the location of zeros.

A matrix A is called reducible if for some permutation matrix P P^TAP has the special form

$$\begin{bmatrix} A_1 & B \\ 0 & A_2 \end{bmatrix},$$

where A_1 and A_2 are square matrices.

A matrix is irreducible iff it is not reducible.

Consider the matrix

$$\begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

This matrix is not reducible.

How to prove this?

First, we need to understand permutation matrices.

Recall: Standard basis for $\mathbb{R}^n$:

$\{e^1, e^2, \ldots, e^n\}$

Here e^i is the vector in which all the entries are zeroes except the i-th entry. The i-th entry is 1.

A matrix P is a permutation matrix iff:

- Each column of P must be an element in the standard basis.
- P is orthogonal.

Example: $[e^2, e^1, e^3 \ldots, e^n]$ is a permutation matrix.

- Let V be an $n \times n$ matrix.
- Let $P = [e^n, e^2, e^3, \ldots, e^{n-1}, e^1]$.
- Write $V = [v^1, v^2, \ldots, v^n]$.
- Then, $VP = [v^n, v^2, v^3, v^4, \ldots, v^{n-1}, v^1]$.
- Right multiplication of a permutation matrix rearranges the columns of V.
- Left multiplication of a permutation matrix rearranges the rows of V.
- $P^T VP$ will be simultaneous rearrangement of rows and columns of V.

Consider the matrix

$$A = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Take $P = [e^2, e^1, e^3, \ldots, e^n]$.

$$AP = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

$P^T AP$ will be

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

This matrix is not in the special form.

Take $P = [e^3, e^2, e^1, \ldots, e^n]$.

$$AP = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

$P^T AP$ will be

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

This matrix is not in the special form.

To complete the proof, we need to verify for all permutation matrices.

How many 4×4 permutation matrices are there?

$1 \times 2 \times 3 \times 4 = 24$

Not easy to verify.

Draw the digraph of A

A is irreducible iff digraph of A is strongly connected.

Non-negative Irreducible matrices and positive matrices are related.

Let A be an $n \times n$ irreducible matrix and non-negative.

Then, $(I + A)^{n-1} > 0$.

In fact, given any n-positive numbers, $k_0, k_1, \ldots, k_{n-1}$,

$$k_0 I + k_1 A + k_2 A^2 + \cdots + k_{n-1} A^{n-1} > 0.$$

We will use this fact to prove Frobenius theorem.