Perron's Theorem (Contd..)

December 5, 2018

Theorem (Perron Frobenius for cones)

Let A be an $n \times n$ real matrix, $n > 1$. Suppose $K \subseteq \mathbb{R}^n$ is a proper cone. Let $AK \subseteq K$. Then,

- *There exists $x \in \mathbb{R}^n$ such that*

$$x \neq 0, \ \ x \in K, \ \ \text{and} \ \ Ax = \rho(A)x.$$

- *Furthermore, there exists $y \in \mathbb{R}^n$ such that*

$$y \neq 0, \ \ y \in K^*, \ \ A^t y = \rho(A)y.$$

We prove the above theorem by assuming that A is diagonalizable.

Given:

- A is an $n \times n$ real matrix.
- $AK \subseteq K$, where $K \subseteq \mathbb{R}^n$ is a proper cone.

To Prove:

- $\rho(A)$ is an eigenvalue of A.

- A real matrix may not have real eigenvalues.
- But an $n \times n$ real matrix has n-complex eigenvalues.

Before we start the proof, we shall see the above statements in detail.

Put

$$V := \mathbb{R}^n + i\mathbb{R}^n.$$

If $x \in \mathbb{C}^n$ (i.e. $x \in V$) then write

$$\text{Real part}(x) \;\text{ by }\; x'$$

and

$$\text{Imag. part}(x) \;\text{ by }\; x''.$$

Define

$$\tilde{A} : V \to V$$

by

$$\tilde{A}(x) = A(x') + iA(x^{''}).$$

Show that for any vector $y \in \mathbb{C}^n$,

$$\tilde{A}y = Ay.$$

$\tilde{A}$ is a complex matrix.

Any $n \times n$ complex matrix has n-eigenvalues.

These numbers are the eigenvalues of A.

A is the given matrix.

We consider $\tilde{A}$.

Define $\tilde{K} := K + iK$.

- $\mathrm{int}(\tilde{K}) = \mathrm{int}(K) + i\,\mathrm{int}(K)$.
- Note:

$$\begin{aligned} \tilde{A}(\tilde{K}) &= A(K) + iA(K) \\ &\subseteq K + iK \\ &= \tilde{K}. \end{aligned} \tag{1}$$

• Let $x, y \in \tilde{K}$. Then, $x + y \in \tilde{K}$.

• If $x \in \tilde{K}$ and if $\alpha \geq 0$, then $\alpha x \in \tilde{K}$.

Let $x \in \tilde{K}$ and $-x \in \tilde{K}$. Then, $x = 0$.

This means that $\tilde{K}$ is closed, convex and pointed.

Claim: $\tilde{A}^2 x = A^2 x' + iA^2 x''$.

$\tilde{A}\tilde{A}(x) = \tilde{A}(A(x' + iAx'')$.

Let $y := Ax' + iAx''$.

Then, $y' = Ax'$ and $y'' = Ax''$.

$\tilde{A}y = Ay' + iAy''$.

$Ay' = A^2 x'$ and $Ay'' = A^2 x^{''}$.

So, $\tilde{A}^2 x = A^2 x' + iA^2 x''$.

For any positive integer r, we have

$$\tilde{A}^r x = A^r x' + iA^r x''.$$

Now

$$\begin{aligned} \tilde{A}^r(\tilde{K}) &= A^r K + iA^r(K) \\ &\subseteq K + iK \\ &= \tilde{K}. \end{aligned} \tag{2}$$

A is diagonalizable.

This means that there exists a basis for $\mathbb{C}^n$, say, $\{x^1, x^2, \ldots, x^n\}$ such that

$$\tilde{A}(x^1) = \lambda_1 x^1,$$
$$\tilde{A}(x^2) = \lambda_2 x^2$$
$$\vdots$$
$$\tilde{A}(x^n) = \lambda_n x^n.$$

To this end, we have the following:

- $\tilde{A} : \mathbb{C}^n \to \mathbb{C}^n$ is a linear map.
- $\tilde{K}$ is a proper cone in V.
- $\tilde{A}\tilde{K} \subseteq \tilde{K}$, $\tilde{A}^2\tilde{K} \subseteq \tilde{K}$, $\tilde{A}^3\tilde{K} \subseteq \tilde{K}$,...
- There exists $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{C}$ and a basis $\{x^1, \ldots, x^n\}$ for $\mathbb{C}^n$ such that

$$\tilde{A}x^i = \lambda_i x^i.$$

We now claim the following:

Lemma

There exists $y \in \mathrm{int}(\tilde{K})$ *such that*

$$y = \alpha_1 x^1 + \alpha_2 x^2 + \cdots + \alpha_n x^n,$$

where $\alpha_i \neq 0$ *for all* i*.*

Proof:

$\mathrm{int}(K) \neq \emptyset$.

Let $v \in \mathrm{int}(\tilde{K})$.

v must be a linear combination of $x^1, \ldots, x^n$.

Let $\beta_1, \beta_2, \ldots, \beta_n \in \mathbb{C}$ be such that

$$v = \beta_1 x^1 + \beta_2 x^2 + \ldots + \beta_n x^n.$$

If the lemma is not true, then some $\beta_i = 0$.

WLOG, let $\beta_1, \beta_2, \ldots, \beta_k = 0$.

So,

$$v = \beta_{k+1} x^{k+1} + \cdots + \beta_n x^n.$$

Let $w := x^1 + \cdots + x^k$.

$v \in \mathrm{int}(\tilde{K}) \;\Rightarrow\; v + \epsilon w \in \mathrm{int}(\tilde{K})$ for some $\epsilon > 0$.

Put $x = v + \epsilon w$.

Now, $x \in \mathrm{int}(\tilde{K})$ is the desired vector.

Proof of the lemma is complete.

First we shall see how to complete the proof of the main theorem in the simplest possible case.

Suppose the eigenvalues λ_i satisfy the following property:

$$|\lambda_1| = \rho, \ \ |\lambda_1| > |\lambda_i| \ \ \forall i.$$

Recall:

$$\tilde{A}x^i = \lambda_i x^i.$$

So, for all $r = 1, 2, \ldots,$

$$\tilde{A}^r x^i = \lambda_i^r x^i.$$

Let $y \in \mathrm{int}(\tilde{K})$ be such that

$$y = \alpha_1 x^1 + \alpha_2 x^2 + \ldots + \alpha_n x^n,$$

where each $\alpha_i \neq 0$.

We know that such a vector exists (Why?).

Compute $\tilde{A}^r y$.

$$\tilde{A}^r y = \alpha_1 \lambda_1^r x^1 + \alpha_2 \lambda_2^r x^2 + \ldots + \lambda_n^r x^n.$$

Now compute the limit of the sequence:

$$\{\tilde{A}^r(\frac{y}{\rho^r})\}_{r=1}^{\infty}$$

Consider the sequences

$$\{(\frac{\lambda_i}{\rho})^r\}_{r=1}^{\infty}.$$

These sequences are bounded.

Can we say that these sequences are convergent? (Why?)

To proceed further, we need the following theorem

Theorem (Bolzano Weirstrass)

Let $\{x^m\}$ be a bounded sequence in $\mathbb{R}^n$ or $\mathbb{C}^n$. Then, some subsequence of $\{x^m\}$ is convergent.

Example

Consider the sequence $\{(i^n, 1)\}$.

Then find a subsequence that is convergent.

WLOG, we can assume that

$$\{(\frac{\lambda_i}{\rho})^r\}_{r=1}^{\infty}.$$

are convergent sequences.

If $i \geq 2$, then these sequences converge to 0.

Let

$$\delta = \lim_{r\to\infty} \frac{\lambda_1^r}{\rho^r}.$$

It is easy to see that

$$\lim_{r\to\infty} \tilde{A}^r(\frac{y}{\rho^r}) = \alpha_1 \delta x^1.$$

$$|\frac{\lambda_1^r}{\rho^r}| = 1 \ \ \forall r \quad \Rightarrow \quad |\delta| = 1.$$

$\alpha_1 \neq 0$ (Since each $\alpha_i \neq 0$).

So,

$$\alpha_1 \delta \neq 0.$$

Put $u := \alpha_1 \delta x^1$.

Now we know that

$$u \neq 0.$$

Claim: $u \in \tilde{K}$.

We have

$$\frac{y}{\rho^r} \in \mathrm{int}(\tilde{K}),$$

and

$$\tilde{A}^r(\tilde{K}) \subseteq \tilde{K} \ \ \forall \ \ r.$$

So,

$$\tilde{A}^r(\frac{y^r}{\rho}) \in \tilde{K} \ \ \forall \, r.$$

• $u := \alpha_1 \delta x^1$ is a limit point of $\tilde{K}$.

So, u is an element of $\tilde{K}$.

$\tilde{A}(u) = \lambda_1 u$.

(Proof: $A(u) = A(\alpha_1 \delta x^1) = \alpha_1 \delta A(x^1) = \alpha_1 \delta \lambda_1 x^1 = \lambda_1 u$.)

Suppose λ_1 is $a + ib$. where $b \neq 0$ or $\lambda_1 \leq 0$.

Then there exist $\gamma_0, \gamma_1, \ldots, \gamma_p > 0$ such that

$$\gamma_0 + \gamma_1 \lambda_1 + \gamma_2 \lambda_1^2 + \ldots + \gamma_p \lambda_1^p = 0.$$

So,

$$\gamma_0 u + \gamma_1 \lambda_1 u + \gamma_2 \lambda_1^2 u + \ldots + \gamma_p \lambda_1^p u = 0. \tag{3}$$

• $\tilde{A}^i u = \lambda_1^i u$.

Also, $u \in \tilde{K}$.

So, $\tilde{A}^i(u) \in \tilde{K}$.

Each $\gamma_i > 0$.

Equation (3) is the following:

$\gamma_0 u + \gamma_1 \tilde{A} u + \gamma_2 \tilde{A}^2 u + \ldots{}_+ \gamma_p \tilde{A}^i u = 0.$

Note:

$$\gamma_0 u \in \tilde{K}, \;\; \gamma_1 \tilde{A} u \in \tilde{K}, \quad \ldots, \gamma_p \tilde{A}^i u \in \tilde{K}$$

Thus, $u = 0$.

This is a contradiction since u is non-zero.

So, $\lambda_1 > 0$.

Thus, $\lambda_1 = \rho$.

Now $\tilde{A}u = \rho u$.

We claim that $Ax = \rho x$ for some $0 \neq x \in K$.

To this end,

$$\tilde{A}u = \rho u, \;\; 0 \neq u \in \tilde{K}.$$

$u = u^{'} + iu^{''}$.

Now $u^{'} \in K$ and $u^{''} \in K$.

$$\begin{aligned} \tilde{A}(u) &= A(u') + iA(u'') \\ &= A(u' + iu'') \\ &= \rho(u' + iu^{''}). \end{aligned} \tag{4}$$

Equating real and imaginary parts,

$$Au' = \rho u' \;\; \text{and} \;\; Au^{''} = \rho(u^{''}).$$

Both u' and $u^{''}$ cannot be zero simultaneously.

The proof is complete.

Now consider the general case:

A is diagonalizable

Eigenvalues of A are:

$$|\lambda_1| = |\lambda_2| = \ldots = |\lambda_k| > |\mu_1| \geq |\mu_2| \geq \ldots |\mu_s|.$$

Recall: $\tilde{A} : V \to V$ is defined by

$$\tilde{A}(x + iy) = Ax + iAy.$$

Let $\tilde{A}x^i = \lambda_i x^i$ for all $i = 1 : k$

Let $\tilde{A}v^i = \mu_i v^i$ for all $i = 1 : s$.

CLAIM: There exists a non-trivial linear combination of $x^1, x^2, \ldots, x^k$ belonging to $\tilde{K}$.

Let $y \in \tilde{K}$ be such that

$$y = \alpha_1 x^1 + \ldots + \alpha_k x^k + \beta_1 v^1 + \ldots + \beta_s v^s,$$

where each $\alpha_i \neq 0$.

Apply $\tilde{A}^r$ on both the sides.

Divide by ρ.

$$\lim_{r\to\infty} \tilde{A}^r(\frac{y}{\rho^r}) = \delta_1\alpha_1 x^1 + \delta_2 x^2 + \ldots + \alpha_k\delta_k x^k.$$

Here each $|\delta_i| = 1$.

RHS $\neq 0$.

$\tilde{K}$ is closed in V.

$$\frac{y}{\rho^r} \in \tilde{K} \ \ \forall r.$$

$$\Rightarrow \ \tilde{A}^r\left(\frac{y}{\rho^r}\right) \subseteq \tilde{K} \ \ \forall r.$$

$$\Rightarrow \ u := \delta_1 \alpha_1 x^1 + \delta_2 x^2 + \ldots + \alpha_k \delta_k x^k$$

is a limit point of $\tilde{K}$.

We now have the following claim:

CLAIM:

Suppose $\lambda_j = a + ib$, where $b \neq 0$. Then,

$$\mathrm{span}\{x^1, x^2, \ldots, x^{i-1}, x^{i+1}, \ldots, x^n\} \cap K \neq \{0\}.$$

Proof of the claim

WLOG, $\lambda_j = \lambda_k$.

Then there exist $c_0, c_1, c_2, \ldots, c_p > 0$ such that $c_0 + c_1\lambda_k + c_2\lambda_k^2 + \ldots + c_p\lambda_k^p = 0$.

$$u = \delta_1\alpha_1 x^1 + \delta_2 x^2 + \ldots + \alpha_k\delta_k x^k \in \tilde{K}.$$

Define $\beta_i := \alpha_i\delta_i$.

Then,

$$u = \beta_1 x^1 + \beta_2 x^2 + \ldots + \beta_k x^k.$$

Now consider the vector

$$\widetilde{x} := \beta_k (c_0 x^k + c_1 \lambda_k x^k + c_2 \lambda_k^2 x^k + \ldots + c_p \lambda_k^p x^k)(\textit{This is } 0)$$

$$+$$

$$\beta_1 (c_0 x^1 + c_1 \lambda_1 x^1 + c_2 \lambda_1^2 x^1 + \ldots + c_p \lambda_1^p x^1)$$

$$+$$

$$\vdots$$

$$+$$

$$\beta_{k-1} (c_0 x^{k-1} + c_1 \lambda_1 x^{k-1} + c_2 \lambda_1^2 x^{k-1} + \ldots + c_p \lambda_1^p x^{k-1}).$$

Note:

$\widetilde{x}$ is a linear combination of $x^1, x^2, \ldots, x^{k-1}$.

We will now prove that

- $\widetilde{x} \in \tilde{K}$.

To do this

- We compute the coefficients $c_0, c_1, \ldots, c_p$ of $\widetilde{x}$ explicitly.

The coefficient of c_0 in $\widetilde{x}$:

$$\sum_{i=1}^{k} \beta_i x^i$$

which is u.

The coefficient of c_1 in $\widetilde{x}$:

$$\sum_{i=1}^{k} \lambda_i \beta_i x^i.$$

But $Ax^i = \lambda_i x^i$.

So,

$$\begin{aligned}\sum_{i=1}^{k} \lambda_i \beta_i x^i &= \sum_{i=1}^{k} \beta_i A x^i. \\ &= \sum_{i=1}^{k} A(\beta_i x^i). \\ &= Au.\end{aligned} \tag{5}$$

Co-efficient of c_2 in $\widetilde{x}$:

$$\sum_{i=1}^{k} \lambda_i^2 \beta_i x^i.$$

This is $A^2 u$.

To sum up:

$$\widetilde{x} := c_0 u + c_1 \tilde{A} u + \ldots + c_p \tilde{A}^p u.$$

$$u \in \tilde{K}$$

and so,

$$\tilde{A}^r(u) \in \tilde{K} \ \ \forall r.$$

All $c_i > 0$.

So, $\widetilde{x} \in \tilde{K}$.

Also, note that $\widetilde{x}$ is not zero. (Why?)

We already know the following:

- $\widetilde{x}$ is a linear combination of $x^1, \ldots, x^{k-1}$.

To summarize:

If λ_k is not positive, then we can find a vector $\widetilde{x}$ satisfying the following properties:

1. $\widetilde{x}$ is a linear combination of $x^1, \ldots, x^{k-1}$.
2. $\widetilde{x} \in \tilde{K}$.
3. $\widetilde{x} \neq 0$.

Suppose all of the following are not positive:

$$\lambda_2, \lambda_3, \ldots, \lambda_k.$$

Apply the previous argument and conclude that

$x^1 \in \tilde{K}$.

If λ_1 is not positive, then there exist $\gamma_0, \gamma_1, \ldots, \gamma_p > 0$ such that

$$\gamma_0 + \gamma_1 \lambda_1 + \gamma_2 \lambda_1^2 + \ldots + \gamma_p \lambda_1^p = 0.$$

$$\gamma_0 x^1 + \gamma_1 \lambda_1 x^1 + \gamma_2 \lambda_1^2 x^1 + \ldots + \gamma_p \lambda_1^p x^1 = 0.$$

This equation is same as:

$$\gamma_0 x^1 + \gamma_1 \tilde{A} x^1 + \gamma_2 \tilde{A}^2 x^1 + \ldots + \gamma_p \tilde{A}^r x^1 = 0.$$

$\gamma_0 x^1 \in \tilde{K} \;\; , \gamma_j \tilde{A}^j x^j \in \tilde{K}$.

So, $x^1 = 0$ which is not possible.

So, $\tilde{A} x^1 = \rho x^1$.

Some of the following are positive and non-positive:

$\lambda_1, \lambda_2, \ldots, \lambda_k$.

WLOG, let $\lambda_1, \ldots, \lambda_m$ be positive and the remaining be non-positive.

We can find a vector $\widetilde{x}$ such that

1. $\widetilde{x}$ is a linear combination of $x^1, \ldots, x^m$.
2. $\widetilde{x} \in \tilde{K}$.
3. $\widetilde{x} \neq 0$.

In this case,

$$\lambda_1 = \lambda_2 = \ldots = \lambda_m = \rho$$

Let

$$\widetilde{x} = \eta_1 x^1 + \eta_2 x^2 + \ldots + \eta_m x^m.$$

$$\tilde{A}\widetilde{x} = \eta_1 A x^1 + \eta_2 A x^2 + \ldots + \eta_m A x^m.$$

$$\begin{aligned} \tilde{A}\widetilde{x} &= \eta_1 \lambda_1 x^1 + \eta_2 \lambda_2 x^2 + \ldots + \eta_m \lambda_m x^m \\ &= \rho(\eta_1 x^1 + \eta_2 x^2 + \ldots + \eta_m x^m) \\ &= \rho \widetilde{x}. \end{aligned} \tag{6}$$

To this end, we have shown that

$\tilde{A}u = \rho u$ for some $0 \neq u$ in $\tilde{K}$.

Now

$$\begin{aligned} \tilde{A}(u) &= Au' + iAu'' \\ &= \rho u \\ &= \rho(u' + iu''). \end{aligned} \tag{7}$$

$$Au' = \rho u' \text{ and } Au'' = \rho u''.$$

$u' \in K$ and $u'' \in K$.

Both of them cannot be zero.

This proves the theorem.

Note that we have proved the following lemma:

Lemma (*)

Suppose $\tilde{A}$ is $n \times n$. Let $x^1, \ldots, x^k \in \mathbb{C}^n$ be eigenvectors in $\tilde{K}$. Let

$$\tilde{A}x^i = \lambda_i x^i \ \ (i = 1 : k).$$

Then at least one $\lambda_i > 0$. If some λ_j is complex or negative real number, then there exists $v \in \tilde{K}$ such that v is a linear combination of $x^1, x^2, \ldots, x^{j-1}, x^{j+1}, \ldots, x^k$.

Definition

Let $K \subseteq \mathbb{R}^n$ be a proper cone. We say that A is K-positive if

$$x \in K \setminus \{0\} \quad \Rightarrow \quad Ax \in \mathrm{int}(K).$$

Theorem

Suppose A is K-positive. Then the following items hold:

1. $$Ax = \lambda x, \;\; 0 \neq x \in K \quad \Rightarrow \quad x \in \mathrm{int}(K).$$
2. *If $Ax = \lambda_1 x,$ and $Ay = \lambda_2 y$ and $x, y \in K$, then $x = \alpha y$ for some α.*

Proof:

Let $Ax = \lambda x$ where $x \in \partial K$. Then $y^* Ax = 0$ for some $0 \neq y \in K$.

However this contradicts that $Ax \in \mathrm{int}(K)$.

This proves 1.

To prove 2, we proceed as follows:

By 1, $x, y \in \mathrm{int}(K)$.

But this will imply that A has a eigenvector in the boundary of the cone K.

Again apply 1 and get a contradiction.