Cones in a finite dimensional euclidean space

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Let X be a real vector space.

If it has a basis, it is called finite dimensional.

(This means that there exist $v_1, \ldots, v_n$ in X such that $v_1, \ldots, v_n$ are linearly independent and

$$\operatorname{span}\{v_1, \ldots, v_n\} = X.)$$

Let $\langle ., . \rangle$ be an inner-product defined on X.

An inner-product can always be defined.

Now the distance between any two vectors in X can be defined.

Let $x \in X$.

Then $\|x\| := \sqrt{\langle x, x \rangle}$.

$\|x\|$ is the distance between the zero vector and x.

$\|x - y\|$ is the distance between x and y.

Open balls in X:

Fix a point a in X and $r > 0$.

$$B(a, r) := \{x \in X : \|x - a\| < r\}.$$

- $B(a, r)$ is called the open ball in X with center a and radius r.

- Let $\Omega \subseteq X$.

- Let $p \in \Omega$. Then, p is called an interior point of Ω if there exists an open ball $B(p, r) \subseteq \Omega$.

- If all points of Ω are interior points, then Ω is called an open set in X.

- Examples...

• A set $F \subseteq X$ is called closed if F^c is open.

Recall: $F^c = X \setminus F$.

F is closed iff

$$x_m \in F \ \ (\forall m), \quad \lim_{m\to\infty} x_m = p \ \ \Rightarrow \ \ p \in F.$$

Examples...

Given a set G in X, $\mathrm{int}(G)$ is the subset of G that contains all interior points of G.

$\mathrm{int}(G)$ is an open set in X.

Given any two vectors x and y in $\mathbb{R}^n$, the standard dot product $x^T y$ is an inner-product between x and y.

$\|x\| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{\frac{1}{2}}$.

$\mathbb{R}^n_+$ is closed in $\mathbb{R}^n$.

$\mathrm{int}(\mathbb{R}^n_+) = \mathbb{R}^n_{++}$.

Cones in X

Consider a set $K \subseteq X$ such that $0 \in K$.

We will say that K is a cone if

1. $x \in K$ and $y \in K \;\; \Rightarrow \;\; x + y \in K$.
2. $\alpha \geq 0$ and $x \in K \;\; \Rightarrow \;\; \alpha x \in K$.
3. K is a closed set in X.
4. K has an interior point.
5. If $x \in K$ and $x \in -K$, then $x = 0$.

• Notice that $\mathbb{R}^n_+$ satisfies all the above conditions.

• Hence $\mathbb{R}^n_+$ is a cone.

Lorentz cone

Other names:
Second-order cone, Ice-cream cone

First we write a vector $x = (x_1, \dots, x_n)$ in $\mathbb{R}^n$ as follows:

$$x = (x_1, p), \quad p = (x_2, \dots, x_n).$$

$K_n := \{x \in \mathbb{R}^n : \|p\| \leq x_1\}$.

For example in $\mathbb{R}^3$,

$$K = \{(x, y, z) : y^2 + z^2 \leq x^2, \quad x \geq 0\}.$$

The above definition can be generalized.

Fix $1 < p < \infty$.

If $x \in \mathbb{R}^n$ define

$$g(x) := (|x_1|^p + |x_2|^p| + \cdots + |x_n|^p)^{1/p}.$$

Define

$$K := \{(\alpha, x) \in \mathbb{R} \times \mathbb{R}^n : g(x) \leq \alpha.\}$$

K is a cone in $\mathbb{R}^{n+1}$.

Fix N a positive integer.

Let V be the set of all symmetric real matrices of order $N \times N$.

V is a vector space with usual matrix addition and scalar multiplication.

V is a finite dimensional space.

Its dimension is $N(N+1)/2$.

Let K_N be the set of all positive semidefinite matrices in V.

K_N is a cone in V.

Dual cones

Given a cone K in X, there is a dual cone associated with K. This will be denoted by K^*.

$$K^* := \{v \in X : \langle v, x \rangle \geq 0 \ \ \forall x \in K\}.$$

Self-dual cones:

$$K^* = K.$$

$\mathbb{R}^n_+$, Ice-cream cone and positive semidefinite cone are self-dual cones.

Let Q be an $n-1 \times n-1$ positive definite matrix.

Define

$$k(Q) := \{(t,x) \in R^n : \sqrt{x^T Q x} \leq t\}.$$

Exercise: Write Lorentz cone as $k(Q)$ for some Q.

$k(Q)^* = k(Q^{-1})$.

Unless Q is identity, $k(Q)$ is not self-dual.

Let $L : X \to X$ be an isomorphism. Then, the dual of $L(K)$ is $L^T(K^*)$.

$$L(K) = \{L(x) : x \in K\}.$$

Let L^{-T} be the inverse of the transpose of L.

Claim:

$$L^{-T}(K^*) = L(K)^*.$$

Let $p \in L^{-T}(K^*)$.

We now show that $p \in L(K)^*$. Take any element $q \in L(K)$. Verify that $\langle p, q \rangle \geq 0$.

Write $p = L^{-T}(v)$ for some $v \in K^*$ and $q = L(x)$ for some $x \in K$.

$\langle p, q \rangle = \langle L^{-T}(v), L(x) \rangle = \langle v, x \rangle \geq 0$.

So, $L^{-T}(K^*) \subseteq L(K)^*$.

Let $y \in L(K)^*$.

Claim: $y \in L^{-T}(K^*)$.

$\langle y, L(x) \rangle \geq 0$ for all $x \in K$.

So, $L^T(y) \in K^*$.

So, $L^T(y) = x$ for some $x \in K^*$.

This means that $y = L^{-T}(x)$ for some $x \in K^*$.

In other words, $y \in L^{-T}(K^*)$.

$k(Q)$ is an isomorphic image of the Lorentz cone.

Define

$$A = \begin{bmatrix} 1 & 0 \\ 0 & Q^{1/2} \end{bmatrix}.$$

AK_n is $k(Q)$.

Now computing dual is easy.

An exercise

Let K be a cone in X.

Fix a basis for X.

$\{v_1, \ldots, v_n\}$.

Now show that there is an element $y \in K$ such that

$$y = c_1 v_1 + \ldots + c_n v_n$$

where each $c_i \neq 0$.

Let $y \in \operatorname{int} K$.

After all y is a vector, so a linear combination of $v_1, \ldots, v_n$.

Write $y = \alpha_1 v_1 + \cdots + \alpha_n v_n$.

Some α_i may be 0.

WLOG let $0 = \alpha_{k+1} = ... = \alpha_n$.

Now, use the fact that $y \in \operatorname{int}(K)$ and $\operatorname{int}(K)$ is an open set.

For some $r > 0$, $B(y, r) \subseteq \operatorname{int}(K)$.

Define $z := y + \frac{r}{2} \frac{v_{k+1}}{\|v_{k+1}\|}$.

$z \in B(y, r)$.

Note:

$y = \alpha_1 v_1 + \cdots + \alpha_k v_k$.

$z = \alpha_1 v_1 + \cdots + \alpha_k v_k + \beta_{k+1} v_{k+1}$; $\beta_{k+1} \neq 0$.

$z \in B(y, r) \subseteq K$.

Continue this until all the coefficients are non-zero.

In the end, we get vector in K such that the coefficients are non-zero.

Let K be a cone in X. Let $x \in K$. Then, we say that $x_K \in K$ is a projection of x onto K if

1. $x_K \in K$.
2. $y \in K \;\; \Rightarrow \;\; \|y - x\| \geq \|x_K - x\|$.

Given an element $x \in X$, we do not know whether there is a projection.

If there is a projection of some element y in K, is there any other vector in K which is a projection of y?

We will address the above items.

Let $x \in X$ be given.

CLAIM: Suppose a vector y satisfy the following:

$$\langle y - x, k - y \rangle \geq 0 \ \ \forall\, k \in K.$$

Then $\|x - y\| \leq \|x - k\|$ for all $k \in K$.

Verify the identity:

$$\|x - k\|^2 = \|x - y\|^2 + \|y - k\|^2 + 2\langle y - x, k - y \rangle.$$

Proof of the claim is complete.

If x_K is a projection of x, then

$$\langle x_K - x, k - x_K \rangle \geq 0 \ \ \forall\, k \in K.$$

PROOF:

Let $0 \neq k \in K$.

Fix $\alpha \in (0, 1)$.

Define

$$x_\alpha := (1 - \alpha) x_K + \alpha k.$$

$x_\alpha \in K$ (Why?)

Verify the identity

$$2\langle x_K - x, k - x_K \rangle = -\alpha \|k - x_K\|^2 + \frac{1}{\alpha}[\|x - x_\alpha\|^2 - \|x - x_K\|^2].$$

Second term is non-negative (Why?)

$2\langle x_K - x, k - x_K \rangle \geq -\alpha \|k - x_K\|^2$

This is valid for all α.

The proof is complete.

CLAIM: There exists at most one projection.

First we will verify this

If x and y belong to K, then $\|x_K - y_K\| \leq \|x - y\|$.

Put $\delta := y - x$ and $\delta_K = y_K - x_K$.

We now know that $\langle x_K - x, y_K - x_K \rangle \geq 0$.

$\langle y - y_K, y_K - x_K \rangle \leq 0$.

Add the above two.

$\langle \delta - \delta_K, \delta_K \rangle \geq 0$.

Now the proof is complete from

$$\|\delta\|^2 = \|\delta - \delta_K\|^2 + \|\delta_K\|^2 + 2\langle \delta - \delta_k, \delta_k \rangle \geq \|\delta_K\|^2.$$

Take $x = y$ and apply the above inequality to see that there is at most one projection.

CLAIM:

Given a cone K, for every vector x there is a projection.

Consider a vector $x \in X$.

Define $\delta := \inf\{\|x - k\| : k \in K\}$.

This number is well-defined (Why)

There exists a sequence $\{s_n\}$ in K such that

$$\|x - s_n\| \to \delta.$$

Use the identity

$$\|s_m - s_n\|^2 = 2\|x - s_m\|^2 + 2\|x - s_n\|^2 - 4\|x - (s_m + s_n)/2\|^2.$$

So,

$$\|s_m - s_n\|^2 \leq 2\|x - s_m\|^2 + 2\|x - s_n\|^2 - 4\delta^2.$$

This says that s_n is Cauchy.

So,s_n converges.

Let $s_n \to p$.

K is closed.

So, $p \in K$.

Proof is complete (Since $p = x_K$).

Let $x \in X$.

Consider $x - x_K$

Then, $\langle x_K - x, k - x_k \geq 0$ for all $k \in K$.

The above inequality tells that $x_K - x \in K^*$ (Why?)

So, $x = y - z$ where $y \in K$ and $z \in K^*$.

Show that $\langle y, z \rangle = 0$.

Now the following can be shown: $x = y - z$ where $y \in K$ and $z \in K^*$ implies

$$y = x_K \;\text{ and }\; z = x_{K^*}.$$

CLAIM: $K = K^{**}$.

$K \subseteq K^{**}$ follows by definition.

Let $x \in K^{**}$.

Now

$$x = x_K - x_{K^*}; \quad x = x_{K^*} - x_{K^{**}}.$$

$x_{K^{**}} = x$. So, $2x = x_{K^*}$. Add the two

$2x = x_K - x_{K^{**}}$.

$3x = x_K$.

Hence, $x \in K$.