complex numbers

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Suppose we want to show that for a complex number z,

$$c_0 + c_1 z + c_2 z^2 + \cdots + c_p z^p = 0 \qquad (*)$$

for some $c_0, c_1, \ldots, c_p > 0$.

Consider a polynomial

$$g(z) = \alpha_0 + \alpha_1 z + \cdots + \alpha_m z^m$$

such that each $\alpha_i > 0$.

In case $k_0 + k_1 g + k_2 g^2 + \cdots + k_q g^q = 0$ for some positive $k_i' s$.

Then $(*)$ will be true.

Suppose $z = a + ib$, where $b \neq 0$.

Then, $\mathrm{R.part}(z^{n_0}) \leq 0$ for some n_0.

Suppose n_0 is such that $\mathrm{R.part}(z^{n_0}) < 0$. (Case 1)

WLOG, let n_0 be the least integer satisfying

$$\mathrm{R.part}(z^{n_0}) < 0.$$

Suppose $n_0 = 1$.

Then, $z = -a + ib$ where $a > 0$.

Now, $w := g(z) = a + z = ib$.

Now, there exist $c_0, c_1, \ldots, c_p > 0$ such that

$$c_0 + c_1 g + c_2 g^2 + \cdots + c_p g^p = 0.$$

g is a polynomial with positive coefficients.

Hence there exist $k_0, k_1, \ldots, k_q > 0$ such that

$$k_0 + k_1 z + k_2 z^2 + \cdots + k_q z^q = 0.$$

Now we need to deal with the case $n_0 \geq 2$.

Note that if $m < n_0$, then $\mathrm{R.part}(z^m) \geq 0$.

Put $\tilde{z} := 1 + z + z^2 + z^3 + \cdots + z^{n_0 - 1}$.

$\mathrm{R.part}(\tilde{z}) > 0$.

Write $z^{n_0} = -a + ib$, $a > 0$.

$$\tilde{z} + cz = ib$$

for some $c > 0$. $(c = ?)$

Define $g(z) := 1 + z + z^2 + \cdots + z^{n_0 - 1} + cz$.

There exist $a_0, a_1, \ldots, a_q > 0$ such that

$$a_0 + a_1 g + a_2 g^2 + \cdots + a_q g^q = 0.$$

Hence there exist $\beta_0, \beta_1, \ldots, \beta'_q > 0$ such that

$$\sum_{i=0}^{q'} \beta_i z^i = 0.$$