Non-diagonalizable case

December 6, 2018

• Let K be a cone in $\mathbb{R}^n$

Let A be an $n \times n$ nilpotent matrix.

Let $AK \subseteq K$. Then, $Ax = \rho x$ for some $x \in K$.

Proof.

Let $0 \neq x \in K$.

If $Ax = 0$, then we are done

Then there exists $r > 1$ a positive integer such that

$$y := A^{r-1}x \neq 0 \text{ and } A^r x = 0.$$

So, $A(y) = 0$.

$0 \neq y \in K$.

The proof is complete. □

- Let A be an $n \times n$ real matrix.
- A is not necessarily diagonalizable.
- $AK \subseteq K$.
- $K \subseteq \mathbb{R}^n$ is a proper cone.

Now we show that there exists $x \in \mathbb{R}^n$ such that

$$Ax = \rho x, \quad x \neq 0, \quad x \in K$$

where ρ is the spectral radius of A.

Consider this simple case:

$$\begin{aligned} Ax^1 &= \lambda_1 x^1 \\ Ax^2 &= x^1 + \lambda_1 x^2 \\ Ax^3 &= \lambda_3 x^3 \\ &\vdots \\ Ax^n &= \lambda_n x^n. \end{aligned} \tag{1}$$

$x^1, x^2, \ldots, x^n$ is a Jordan basis for $\mathbb{C}^n$.

$\lambda_1 = \rho$ and $\rho > |\lambda_i| \ \ \forall i \geq 2$.

- For $x \in \mathbb{C}^n$, define

$$\tilde{A}(x) := Ax' + iAx''.$$

- Let $\tilde{K} := K + iK$ be defined as before.
- We can find $y \in \mathrm{int}(\tilde{K})$ such that

$$y = \alpha_1 x^1 + \alpha_2 x^2 + \ldots + \alpha_n x^n.$$

where each $\alpha_i \neq 0$.

$\tilde{A}x^1 = \lambda_1 x^1.$

$$\begin{aligned} \tilde{A}x^2 &= A(x^{2'}) + iA(x^{2''}) \\ &= A(x^{2'} + ix^{2''}) \\ &= Ax^2 \\ &= x^1 + \lambda_1 x^2 \end{aligned} \tag{2}$$

$$\tilde{A}x^i = \lambda_i x^i \quad i = 3, \ldots, n.$$

We now have

$$\begin{aligned}
\tilde{A}x^1 &= \lambda_1 x^1 \\
\tilde{A}x^2 &= x^1 + \lambda_1 x^2 \\
\tilde{A}x^3 &= \lambda_3 x^3 \\
&\vdots \\
\tilde{A}x^n &= \lambda_n x^n.
\end{aligned} \tag{3}$$

$x^1, x^2, \ldots, x^n$ is a Jordan basis for $\mathbb{C}^n$.

$\lambda_1 = \rho$ and $\rho > |\lambda_i| \ \ \forall i \geq 2$.

Computing $\tilde{A}^2 x^2$:

$$\begin{aligned} \tilde{A}^2 x^2 &= \tilde{A}(\tilde{A}x^2) \\ &= \tilde{A}x^1 + \lambda_1 \tilde{A}x^2 \\ &= \tilde{A}(x^1 + \lambda_1 x^2) \\ &= \lambda_1 x^1 + \lambda_1(x^1 + \lambda_1 x^2) \\ &= \lambda_1 x^1 + \lambda_1 x^1 + \lambda_1^2 x^2 \\ &= 2\lambda_1 x^1 + \lambda_1^2 x^2. \end{aligned} \tag{4}$$

In general for every positive integer r,

$$\tilde{A}^r x^2 = r\lambda_1^{r-1} x^1 + \lambda_1^r x^2.$$

$$y = \alpha_1 x^1 + \alpha_2 x^2 + \ldots + \alpha_n x^n.$$

$$\tilde{A}^r(y) = \alpha_1 \lambda_1^r x^1 + \alpha_2 (r \lambda_1^{r-1} x^1 + \lambda_1^r x^2) + \sum_{j=3}^{n} \lambda_j^r x^j \tag{5}$$

$$\lim_{r \to \infty} \tilde{A}^r\left(\frac{y}{r \rho^r}\right) = \frac{\alpha_2 \delta}{\lambda_1} x^1$$

Put $\beta = \frac{\alpha_2 \delta}{\lambda_1}$.

Now $\beta x^1 \in \tilde{K}$.

Clearly $\beta \neq 0$.

$$\begin{aligned} \tilde{A}(\beta x^1) &= \beta \tilde{A}(x^1) \\ &= \beta \lambda_1 x^1 \end{aligned} \tag{6}$$

Suppose λ_1 is not positive.

Now use the fact:

There exist $c_0, c_1, \ldots, c_p > 0$ such that

$$c_0 + c_1\lambda_1 + c_2\lambda_1^2 + \ldots + c_p\lambda_1^p = 0.$$

Now,

$$c_0 y + c_1 \lambda_1 y + c_2 \lambda_1^2 y + \ldots + c_p \lambda_1^p y = 0.$$

$$c_0 y + c_1 \tilde{A} y + c_2 \tilde{A}^2 y + \ldots + c_p \tilde{A}^p y = 0.$$

As $y \in \tilde{K}$,

$$c_i \tilde{A}^i y \in \tilde{K} \ \ \forall i \geq 0.$$

So, $y = 0$.

Now, $\tilde{A}x^1 = \rho x^1$ and $x^1 \in \tilde{K}$.

This gives

$$Ax^{1'} = \rho x^{1'} \text{ and } Ax^{1''} = \rho x^{1''}.$$

Both $x^{1'}$ and $x^{1''}$ belong to K.

Simultaneously both the vectors cannot be zero.

This completes the proof.

A more general case:

$$\tilde{A}x^1 = \lambda_1 x^1$$
$$\tilde{A}x^2 = \quad x^1 + \lambda_1 x^2$$
$$\tilde{A}x^3 = x^2 + \lambda_1 x^3$$
$$\vdots$$
$$\tilde{A}x^k = x^{k-1} + \lambda_1 x^k$$
$$\tilde{A}y^1 = \lambda_2 y^1$$
$$\tilde{A}y^2 = \lambda_3 y^2$$
$$\vdots$$
$$\tilde{A}y^s = \lambda_s y^s.$$

$|\lambda_1| > |\lambda_j| \quad$ for all $j \geq 2$.

Here we have assumed that

A is similar to J, where

$J = \mathrm{diag}(J_1(\lambda_1), \lambda_2, \ldots, \lambda_s)$,

where J_1 is of order $k \times k$ and $k + s = n$.

- In the previous case, we had $k = 2$.

Already we have seen that

$$\tilde{A}^r x^2 = r\lambda_1^{r-1} x^1 + \lambda_1^r x^2.$$

$$\begin{aligned}\tilde{A}^2 x^3 &= \tilde{A}x^2 + \lambda_1 \tilde{A}x^3 \\ &= x^1 + \lambda_1 x^2 + \lambda_1 (x^2 + \lambda_1 x^3) \\ &= x^1 + 2\lambda_1 x^2 + \lambda_1^2 x^3.\end{aligned} \tag{7}$$

Co-efficient: $(1, 2, 1)$

$$\tilde{A}^3 x^3 = \lambda_1^3 x^3 + 3\lambda_1^2 x^2 + 3\lambda_1 x^1.$$

Co-efficient: $(1, 3, 3)$

$$\tilde{A}^4 x^4 = \lambda_1^4 x^3 + 4\lambda_1^3 x^2 + 6\lambda_1^2 x^1.$$

Co-efficient: $(1, 4C_1, 4C_2)$.
In general:

$$\tilde{A}^r x^3 = \lambda_1^r x^3 + rC_1 \lambda_1^{r-1} x^2 + rC_2 \lambda_1^{r-2} x^1.$$

In general

$$\tilde{A}^r x^k = \lambda_1^r x^k + rC_1 \lambda_1^{r-1} x^{k-1} + rC_2 \lambda_1^{r-2} x^{k-2} + \ldots + rC_{k-1} \lambda_1^{r-k+1} x^1.$$

Let $y \in \operatorname{int} \tilde{K}$ be such that

$$y = \alpha_1 x^1 + \alpha_2 x^2 + \cdots + \alpha_k x^k + \alpha_{k+1} y^1 + \alpha_{k+2} y^2 + \cdots + \alpha_{k+s} y^s.$$

where each $\alpha_i \neq 0$.

$$\tilde{A}^r(y) = \alpha_1 \tilde{A}^r x^1 + \alpha_2 \tilde{A}^r x^2 + \cdots + \alpha_k \tilde{A}^r x^k$$

$$+$$

$$\alpha_{k+1} \tilde{A}^r y^1 + \ldots + \alpha_{k+s} \tilde{A}^r y^s.$$

$$\lim_{r\to\infty} \tilde{A}^r\left(\frac{y}{rC_{k-1}|\lambda_1|^r}\right) = \alpha_k \delta x^1$$

$\alpha_k \delta x^1 \neq 0$ and it is a point in $\tilde{K}$.

Put $w := \alpha_k \delta x^1$. $w \in \tilde{K}$.

w is an eigenvector of $\tilde{A}$.

Eigenvalue corresponding to w is λ_k.

If λ_k is not positive, then we get a contradiction using $(*)$.

Now the proof is complete.

A more general case:

A is similar to a Jordan matrix of the form:

$$\mathrm{diag}(J_{\lambda_1}, J_{\lambda_2}, \ldots, J_{\lambda_\nu}, \lambda'_1, \ldots, \lambda'_s),$$

where

$$|\lambda_1| = |\lambda_2| = \ldots = |\lambda_\nu| > \lambda'_1 > \lambda'_2 > \ldots \lambda'_s;$$

J_{λ_i} has order k_i;

$k_1 + k_2 + \ldots + k_\nu + s = n$.

Assume that

$$\begin{aligned}
\tilde{A}x^{1,1} &= \lambda_1 x^{1,1} \\
\tilde{A}x^{1,2} &= x^{1,1} + \lambda_1 x^{1,2} \\
\tilde{A}x^{2,1} &= \lambda_2 x^{2,1} \\
\tilde{A}x^{2,2} &= x^{2,1} + \lambda_2 x^{2,2} \\
\tilde{A}x^{5} &= \lambda_5 x^{5} \\
&\vdots \\
\tilde{A}x^{n} &= \lambda_n x^{n}.
\end{aligned}$$

Put $x^1 = x^{1,1}$ and $y^1 = x^{1,2}$.

Put $x^2 = x^{2,2}$ and $y^2 = x^{2,2}$

Now

$$\tilde{A}x^1 = \lambda_1 x^1, \;\; \tilde{A}y^1 = x^1 + \lambda_1 y^1$$

$$\tilde{A}x^2 = \lambda_2 x^2, \;\; \tilde{A}y^2 = x^2 + \lambda_2 y^2$$

$$\tilde{A}x^i = \lambda_i x^i \;\; (i = 5 : n).$$

Already we have seen that,

$$\tilde{A}^r y^1 = \lambda_1^r y^1 + rC_1 \lambda_1^{r-1} x^1.$$

Let $y \in \tilde{K}$ be such that

$$y = \alpha_1 x^1 + \alpha_{1'} y^1 + \alpha_2 x^2 + \alpha_{2'} y^2 + \sum_{i=5}^{n} \alpha_i x^i,$$

where each $\alpha_i \neq 0$. Clearly

$$\tilde{A}^r\left(\frac{y}{rC_1\rho^r}\right) \to \alpha_1 x^1 + \alpha_2 x^2.$$

Put $u = \alpha_1 x^1 + \alpha_2 x^2$.

u is a non-zero vector in $\tilde{K}$.

Important observation: It is a linear combination of x^1 and x^2. These are eigenvectors.

If λ_2 is not positive, then we can show that $x^1 \in \pm\tilde{K}$. In this case the theorem will then be true.

Suppose both λ_1 and λ_2 are positive.

So, $\lambda_1 = \lambda_2 = \rho$.

Let $u := \alpha x^1 + \beta x^2 \in \tilde{K}$.

Now $\tilde{A}u = \rho u$.

This proves the result.

Let A be similar to the following matrix:

$$\mathrm{diag}(J_{\lambda_1}, \ldots, J_{\lambda_k}, J_{\mu_1}, \ldots, J_{\mu_s})$$

Here $|\lambda_1| = |\lambda_2| = \ldots = |\lambda_k| > |\mu_1| \geq |\mu_2| \geq \ldots \geq |\mu_s|$

Let the order of J_{λ_i} be $f(i)$.

Let the order of J_{μ_i} be $g(i)$.

So, $f(1) + f(2) + \cdots + f(k) + g(1) + \cdots + g(s) = n$.

$$\tilde{A}x^{1,i} = \lambda_i x^{1,i}.$$

$$\tilde{A}x^{2,i} = x^{1,i} + \lambda_i x^{2,i}$$

$$\tilde{A}x^{3,i} = x^{2,i} + \lambda_i x^{3,i}$$

$$\vdots$$

$$\tilde{A}x^{f(i),i} = x^{f(i)-1,i} + \lambda_i x^{f(i),i}.$$

$(i = 1 : k)$

$$\tilde{A}y^{1,i} = \mu_i y^{1,i}.$$

$$\tilde{A}y^{2,i} = y^{1,i} + \mu_i y^{2,i}$$

$$\tilde{A}y^{3,i} = y^{2,i} + \mu_i y^{3,i}$$

$$\vdots$$

$$\tilde{A}y^{g(i),i} = y^{f(i)-1,i} + \mu_i y^{g(i),i}.$$

$i = 1 : s$

Assume that

$$f(1) = f(2) = \cdots = f(\nu) = (\eta \ \text{say}) \quad \nu \leq k.$$

Jordan basis:

Define

$$S_{f(i)} := \{x^{1,i}, x^{2,i}, \ldots, x^{f(i),i}\}$$

$$S_{g(i)} := \{y^{1,i}, y^{2,i}, \ldots, y^{g(i),i}\}$$

$$\cup_{i=1}^{k} S_{f(i)} \cup \cup_{i=1}^{s} S_{g(i)}$$

is a basis for $\mathbb{C}^n$.

We can find a vector $y \in \mathrm{int}(\tilde{K})$ such that

$$y = \sum_{i=1}^{f(1)} \alpha_{i,1} x^{i,1} + \sum_{i=1}^{f(2)} \alpha_{i,2} x^{i,2} + \cdots + \sum_{i=1}^{f(k)} \alpha_{i,k} x^{i,k}$$

$$+$$

$$\sum_{i=1}^{g(1)} \beta_{i,1} y^{i,1} + \sum_{i=1}^{g(2)} \beta_{i,2} y^{i,2} + \cdots + \sum_{i=1}^{g(s)} \beta_{i,s} y^{i,s},$$

where $\alpha_{i,j} \neq 0$ and $\beta_{i,j} \neq 0$ for all i, j.

$$\tilde{A}^r x^{k,i} = \lambda_i^r x^{k,i} + rC_1 \lambda_i^{r-1} x^{k,i-1} + rC_2 \lambda_i^{r-2} x^{k,i-2}$$

$$+ \cdots +$$

$$rC_{k-1} \lambda_i^{r-k+1} x^{k,1}$$

rC_η is a polynomial in r. Its degree is η.

Compute the limit

$$\lim_{r\to\infty} A^r\left(\frac{x^{k,i}}{rC_{k-1}\rho^r}\right) = \delta_i x^{k,1}$$

$$\lim_{r\to\infty} A^r\left(\frac{y^{i,1}}{rC_\eta\rho^r}\right) = 0.$$

$$\lim_{r\to\infty} A^r\left(\frac{y}{rC_\eta\rho^r}\right) = z$$

where z is a linear combination of $x^{1,1}, x^{1,2}, \ldots, x^{1,\nu}$.

Each one of them is an eigenvector.

Now Apply Lemma $(*)$ to complete the proof.