cones (Contd..)

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Lemma

If $AK \subseteq K$ and $K \in \mathbb{R}^n$ is a proper cone, and if A has two linearly independent eigenvectors in $\mathrm{int}(K)$, then A has a eigenvector in the boundary of K. Furthermore, the eigenvalues are equal.

PROOF:

- Let $Ax^1 = \lambda_1 x^1$.
- Let $Ax^2 = \lambda_2 x^2$.
- $x^1 \in \mathrm{int}\, K$ and $x^2 \in \mathrm{int}\, K$ are linearly independent.

For some $\alpha > 0$

$$\alpha x^2 - x^3 \in K.$$

Let $\Omega := \{t \in \mathbb{R} : 0 \leq t \leq \alpha, \;\; tx^2 - x^1 \in K\}$.

$\{t \in \mathbb{R} : tx^2 - x^1 \in K\}$ is a closed set in $\mathbb{R}$.

$[0, \alpha]$ is compact.

So, Ω is compact.

$\min\{t : t \in \Omega\}$ has an optimal solution.

Let t_0 be the solution.

t_0 is not zero.

$t_0 x^2 - x^1 \in K$.

Suppose $t_0 x^2 - x^1 \in \operatorname{int} K$.

Then, $(t_0 - \epsilon)x^2 - x^1 \in K$ and also $t_0 - \epsilon > 0$ for some $\epsilon > 0$.

This is a contradiction, since t_0 is the minimum.

So, $x^3 := t_0 x^2 - x^1 \in \partial K$.

λ_1 and λ_2 cannot be zero.

$$\begin{aligned} Ax^3 &= t_0 \lambda_2 x^2 - \lambda_1 x^1 \\ &= \lambda_1 \left(t_0 \frac{\lambda_2}{\lambda_1} x^2 - x^1\right) \\ &\in K \end{aligned} \tag{1}$$

So,

$$(t_0 \frac{\lambda_2}{\lambda_1} x^2) - x^1 \in K.$$

If $\lambda_2 < \lambda_1$, then this will contradict the minimality of t_0.

So, $\lambda_2 \geq \lambda_1$.

Similarly, $\lambda_1 \geq \lambda_2$.

Thus, $\lambda_1 = \lambda_2$.

$Ax^3 = t_0 Ax^2 - Ax^1$

This is equal to $t_0 \lambda_1 x^2 - \lambda_1 x^1$.

This in turn equals $\lambda_1 x^3$.